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NONLINEAR BENDING OF RECTANGULAR ORTHOTROPIC PLATES

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Abstract—Displacement formulation of the governing field equations for nonlinear behaviour of rectilinearly orthotropic elastic plates are presented. These equations are then used to obtain an approximate solution of a simply supported rectangular plate subjected to a uniformly distributed load. The Galerkin–Dubnov technique is used to obtain the approximate solution. Nonlinear behaviour is studied for some specific degrees of orthotropy.

INTRODUCTION

MANY important problems of structural strength and stability of plates, arising in modern aircraft construction, cannot be adequately analyzed on the basis of the classical theory since the plate deflections experienced are not small in comparison with the plate thickness. In this case the membrane action of the plate must be considered which leads to nonlinear terms in the equations of equilibrium of the plate element. The field equations which include the effect of the membrane stresses were derived by von Karman for elastic isotropic plates under static loadings. Later his theory was extended to include dynamic effects by Herrmann [1] and to orthotropic plates by Rostovtsev [2]. The formulation given by Rostovtsev is in terms of stress function, and has been used by Nowinski [3], Nowinski and Ismail [4] *et al.*, to study several important problems of practical interest. It is interesting to note that the problems treated so far have stress free edge conditions. While it is possible to use Rostovtsev's formulation for problems in which the edges of the plate are immovable, it is more convenient to formulate the problem in terms of displacements. For isotropic elastic plates such formulation can be found in Chu and Herrmann [7] and elsewhere. To the author's knowledge the field equations in terms of displacements for the case of rectilinearly orthotropic elastic plates are unavailable. The purpose of the present paper is to derive these field equations, and to use them to study the nonlinear behaviour of a simply supported rectangular plate exhibiting rectilinear orthotropy, under uniformly distributed load. In obtaining an approximate solution Galerkin–Bubnov method has been used. The results are compared to other known solutions, and the effect of the degree of orthotropy on the nonlinear behaviour is studied for some specific cases.

BASIC EQUATIONS

The principle of minimum potential energy will be used in deriving the governing equations. According to this principle of all possible displacement configurations of an elastic structure the true displacement will be the one for which the total potential energy is minimum. For the case of nonlinear bending of thin plates the total potential energy of the system is the sum of the work done by external forces and the strain energy of bending and the strain energy of stretching of the middle surface of the plate. Here nonlinearity is introduced through the consideration of stretching of the middle surface.

Consider a thin elastic plate of uniform thickness h, exhibiting rectilinear orthotropy, having any arbitrary boundary. The plate is assumed to be oriented such that the elastic axes are parallel to a set of rectangular coordinate axes (Fig. 1). Using the usual notations of u, v and w for displacements in the plane of the plate and transverse directions, respectively, the strains of the middle surface of the plate can be written as†:

$$\begin{aligned} \varepsilon_{xx} &= u_x + \tfrac{1}{2}w_x^2 \\ \varepsilon_{yy} &= v_y + \tfrac{1}{2}w_y^2 \\ \varepsilon_{xy} &= u_y + v_x + w_x w_y \end{aligned} \tag{1}$$

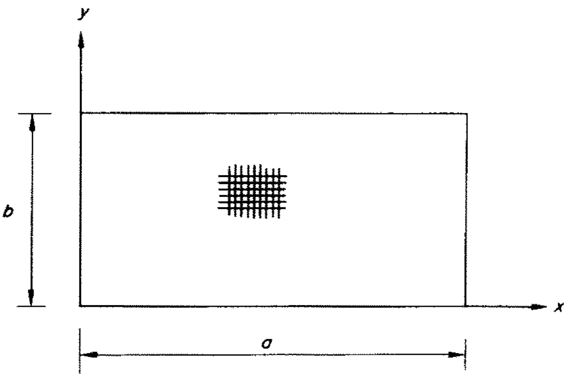

FIG. 1. Geometry of the plate and the directions of the elastic axes.

Hooke's law for orthotropic elastic material is given by:

$$\begin{aligned} \sigma_{xx} &= 1/v_{xy}[E_x\varepsilon_x + v_x E_y \varepsilon_y] \\ \sigma_{yy} &= 1/v_{xy}[E_y\varepsilon_y + v_y E_x \varepsilon_x] \\ \sigma_{xy} &= G\varepsilon_{xy} \end{aligned} \tag{2}$$

where E_x, E_y are Young's modulii in the x and y directions, and G is the shear modulus. And $v_{xy} = 1 - v_x v_y$, where v_x and v_y are the Poisson's ratios in the x and y directions, respectively.

The strain energy of stretching of the middle surface, therefore, can be written as:

$$U^s = \frac{1}{2}\int_V \left\{\frac{\varepsilon_{xx}}{v_{xy}}(E_x\varepsilon_{xx} + v_x E_y \varepsilon_{yy}) + \frac{\varepsilon_{yy}}{v_{xy}}(E_y\varepsilon_{yy} + v_y E_x \varepsilon_{xx}) + \varepsilon_{xy} G \varepsilon_{xy}\right\} \mathrm{d}v. \tag{2}$$

The strain energy of bending of the plate element dA can be written as (see [2]):

$$U^b = \frac{1}{2}\int_A \{D_x w_{xx}^2 + D_y w_{yy}^2 + 2D_x v_y w_{xx} w_{yy} + 4D_k w_{xy}^2\} \, \mathrm{d}A \tag{4}$$

† In what follows, the subscripts associated with u, v and w refer to differentiations, e.g., $u_x = \partial u/\partial x$, $w_{xy} = \partial^2 w/\partial x\, \partial y$, etc.

where w is the transverse displacement of the middle surface of the plate, D_x, D_y are the bending rigidities in x and y directions, and $D_k = Gh^3/12$.

Assume that the plate is resting on an elastic foundation of Winkler type and subjected to a uniformly distributed load q over the entire area of the plate. The work done by the load q and the reaction of the foundation is given by:

$$W = \frac{1}{2}\int_A \{-2qw + kw^2\}\,\mathrm{d}A \tag{5}$$

where k is the foundation modulus.

Combining equations (3–5) and using equations (1), the total potential energy of the system is found to be:

$$\begin{aligned}\Pi = \frac{1}{2}\int_A &\{D_x w_{xx}^2 + D_y w_{yy}^2 + 2D_x v_y w_{xx} w_{yy} + 4D_k w_{xy}^2 + h/v_{xy}[E_x(u_x + \tfrac{1}{2}w_x^2)^2 \\ &+ E_y(v_y + \tfrac{1}{2}w_y^2)^2 + 2v_x E_y(u_x + \tfrac{1}{2}w_x^2)(v_y + \tfrac{1}{2}w_y^2) + Gv_{xy}(u_y + v_x + w_x w_y)^2] \\ &-2qw + kw^2\}\,\mathrm{d}A,\end{aligned} \tag{6}$$

or,

$$\Pi = \int_A f(w, w_x, w_y, w_{xx}, w_{yy}, w_{xy}, u_x, v_y, u_y, v_x)\,\mathrm{d}A. \tag{7}$$

The Euler equations corresponding to the functional given in equation (7) are:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial u_y}\right) = 0 \tag{8i}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial v_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial v_y}\right) = 0 \tag{8ii}$$

$$\frac{\partial f}{\partial w} - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial w_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial w_y}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial f}{\partial w_{xx}}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial f}{\partial w_{yy}}\right) + \frac{\partial^2}{\partial x\,\partial y}\left(\frac{\partial f}{\partial w_{xy}}\right) = 0. \tag{8iii}$$

Using equations (6) and (8), after some rearrangements, we obtain the following equations:

$$u_{xx} + w_x w_{xx} + p^2(u_{yy} + w_x w_{yy}) + (p^2 + v_y)(v_{xy} + w_y w_{xy}) = 0 \tag{9i}$$

$$l^2(v_{yy} + w_y w_{yy}) + p^2(v_{xx} + w_y w_{xx}) + (p^2 + v_y)(u_{xy} + w_x w_{xy}) = 0 \tag{9ii}$$

$$\begin{aligned}w_{xxxx} + 2m^2 w_{xxyy} + l^2 w_{yyyy} - \frac{q}{D_x} + \frac{kw}{D_x} - \frac{12}{h^2}[w_{xx}\{(u_x + \tfrac{1}{2}w_x^2) + v_y(v_y + \tfrac{1}{2}w_y^2)\} \\ + l^2 w_{yy}\{(v_y + \tfrac{1}{2}w_y^2) + v_x(u_x + \tfrac{1}{2}w_x^2)\} + 2p^2 w_{xy}\{u_y + v_x + w_x w_y\}] = 0\end{aligned} \tag{9iii}$$

in which,

$$\begin{aligned}m^2 &= v_y + 2D_k/D_x \\ l^2 &= D_y/D_x = E_y/E_x \\ p^2 &= G/E_x(1 - v_x v_y).\end{aligned} \tag{10}$$

Equations (9) constitute the formulation of the problem in terms of displacements. For isotropic plates $m^2 = l^2 = 1$, $p^2 = (1-\nu)/2$ and the equations reduce to those given by Mansfield [5].

SIMPLY SUPPORTED RECTANGULAR PLATES

Equations (9) will be used to study the nonlinear behaviour of a simply supported rectangular plate (Fig. 1). The edges of the plate are assumed to be immovable. Thus the boundary conditions on u, v and w are:

$$\begin{aligned} &\text{at } x = 0, a: && \text{at } y = 0, b: \\ &u = w = w_{xx} = 0; && v = w = w_{yy} = 0. \end{aligned} \tag{11}$$

The set of equations given in (9) are coupled and nonlinear in nature, and exact solution is extremely difficult to obtain. Therefore, an approximate solution will be obtained here. Let w_0 be the transverse deflection at the centre of the plate (i.e., at $x = a/2$, $y = b/2$), and let

$$w = w_0 \sin \pi x/a \sin \pi y/b. \tag{12}$$

Also, let

$$\begin{aligned} u &= \frac{w_0^2 \pi}{16a}(\cos 2\pi y/b - 1 + \beta^2 \nu_y) \sin 2\pi x/a \\ v &= \frac{w_0^2 \pi}{16b}(\cos 2\pi x/a - 1 + \nu_y/l^2\beta^2) \sin 2\pi y/b \end{aligned} \tag{13}$$

where $\beta = a/b$.

Clearly this choice of the displacements u, v and w satisfy all the boundary conditions given by equation (11). Furthermore, equation (13) satisfies the first two equations of the set of equation (9). To dispose of the last equation of (9) we use the Galkerin–Bubnov procedure, and compute the integral

$$\int_0^a \int_0^b \Lambda \sin \pi x/a \sin \pi y/b \, dx \, dy = 0 \tag{14}$$

where Λ is the left-hand side of equation (9iii). After evaluation of the necessary integrals and some simplification, we obtain the following:

$$a_1 \frac{w_0}{h} + a_2 \left(\frac{w_0}{h}\right)^3 = \frac{qa^4}{D_x h} \tag{15}$$

where,

$$\begin{aligned} a_1 &= \frac{\pi^6}{16}\left(1 + 2m^2\beta^2 + l^2\beta^4 + \frac{ka^4}{\pi^4 D_x}\right) \\ a_2 &= \frac{3\pi^6}{64}\left(3 + 3l^2\beta^4 + 3\nu_x l^2\beta^2 + \beta^2\nu_y - \frac{\nu_y^2}{l^2} - \nu_x\nu_y l^2\beta^4\right). \end{aligned} \tag{16}$$

Since no other solution seems to be available for the present problem it is possible to compare the above result with the isotropic case only for which we have $l = m = 1$, $\nu_x = \nu_y = \nu$, $D_x = D_y = D$, and equation (15) reduces to (with $k = 0$):

$$\tfrac{1}{12}(1+\beta^2)^2\left(\frac{w_0}{h}\right)+\tfrac{1}{16}[4\nu\beta^2+(3-\nu^2)(1+\beta^4)]\left(\frac{w_0}{h}\right)^3 = \frac{16qa^4}{\pi^6Eh^4}(1-\nu^2) \tag{17}$$

which matches exactly with the result given in Ref. [6].

Using equations (1), (2) and (13) the dimensionless membrane stresses $\bar{\sigma}_x$ and $\bar{\sigma}_y$ can be written in the form:

$$\bar{\sigma}_x = \alpha_1\left(\frac{w_0}{h}\right)^2$$
$$\tilde{\sigma}_y = \alpha_2\left(\frac{w_0}{h}\right) \tag{18}$$

where,

$$\bar{\sigma}_x = \frac{8\nu_{xy}a^2\sigma_{xx}}{\pi^2E_xh^2}$$
$$\bar{\sigma}_y = \frac{8\nu_{xy}a^2\sigma_{yy}}{\pi^2E_xh^2}$$
$$\alpha_1 = (2-\beta^2\nu_y+2\beta^2\nu_xl^2-\nu_x\nu_y)$$
$$\alpha_2 = (2l^2\beta^2+\nu_y-\beta^2\nu_y^2). \tag{19}$$

In order to study the effect of the degree of orthotropy on the nonlinear behaviour the following types of orthotropy are considered (Table 1). The data in Table 1 correspond roughly to those concerning two real materials (plywood and delta product).

TABLE 1

Type	E_x	E_y	G	ν_x	ν_y
I	$1{\cdot}0\times10^5$	$0{\cdot}5\times10^5$	$0{\cdot}1\times10^5$	0·05	0·025
II	$1{\cdot}0\times10^5$	$0{\cdot}05\times10^5$	$0{\cdot}05\times10^5$	0·20	0·01
III	E	E	G	0·30	0·30

In Fig. 2 the values of the dimensionless central deflection w_0/h are plotted against the dimensionless load parameter $Q = qa^4/D_xh$ for $\beta = 1, 2$ and for the types of orthotropy given in Table 1. It can be seen from Fig. 2 that for isotropy and weak orthotropy (i.e., Orthotropy I) the effect of the membrane stresses becomes less significant with increase in the aspect ratio. On the other hand, for the case of strong orthotropy (i.e., Orthotropy II) the membrane stresses have a significant effect on the deflection for both values of the aspect ratio. Furthermore, for any given value of β, the central deflection is greatest for strong orthotropy and least for isotropy. It is also clear that the effect of the membrane stresses depends on the combination of the aspect ratio and the degree of orthotropy.

Figure 3 shows the variation of the dimensionless membrane stresses with w_0/h for the orthotropic plates and aspect ratios one and two. It can be seen that the membrane stresses

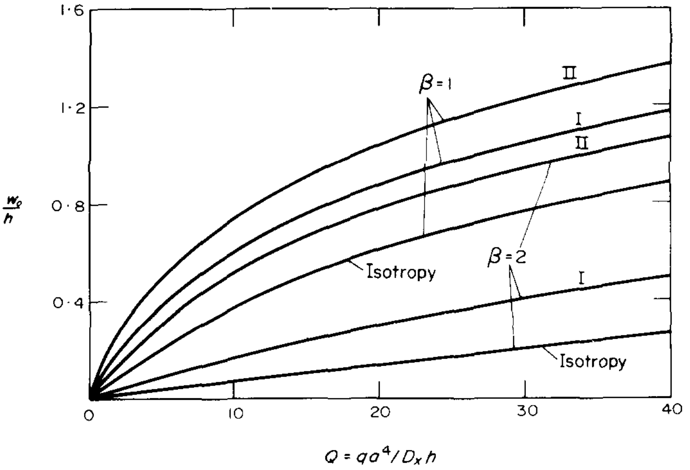

FIG. 2. Load–deflection curves.

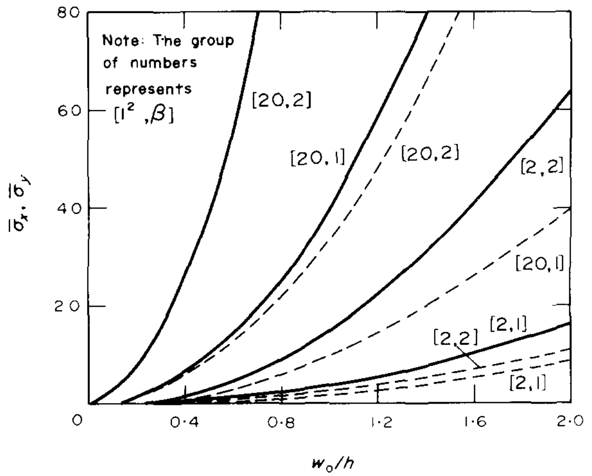

FIG. 3. Plot of membrane stresses vs w_0/h for $\beta = 1$ and $\beta = 2$.

increase rapidly with increasing amplitude. It is obvious that stresses can be either reduced or raised by an appropriate selection of the degree of orthotropy. Due to different tensile moduli, the stresses of a square plate at the centre are not equal to each other and the stress in the y direction associated with the higher tensile modulus is greater than that in the x direction.

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Абстракт—Для определяющих уравнений полей дается изложение в перемещениях, с целью описания нелинейного поведения прямолинейных, ортотропных пластинок. Далее, используются эти уравнения для получения приближенного решения свободно опертой, прямоугольной пластинки, подверженной действию равномерно распределенной нагрузки. Приближенное решение получается путем применения метода Бубнова-Галеркина. Исследуется нелинейное поведение для некоторых специфических порядков ортотропии.